INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2019-20
Statistics - III, Backpaper Examination, January, 2020
Time: 3 Hours **Total Marks: 50**

1. Let $\mathbf{Y} = (Y_1, \dots, Y_n)' \sim N_n(\mathbf{0}, \sigma^2 I_n)$. Find the conditional distribution of $\mathbf{Y}'\mathbf{Y}$ given $\sum_{i=1}^n Y_i$. [5]

2. Consider the model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\mathbf{X}_{n\times p}$ has $\mathbf{1}$ as its first column and $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$.
(a) If $\hat{\beta}$ is the least squares estimator of β, show that $(\hat{\beta} - \beta)'\mathbf{X}'\mathbf{X}(\hat{\beta} - \beta)$ is distributed independently of the residual sum of squares.
(b) Consider the case when there is only one regressor, X_1. When do we have independence of $\hat{\beta}_0$ and $\hat{\beta}_1$?
(c) Find the maximum likelihood estimator of σ^2. Is it unbiased? [12]

3. Consider the following model:
$y_1 = \theta + \gamma + \epsilon_1$
$y_2 = \theta + \phi + \epsilon_2$
$y_3 = 2\theta + \phi + \gamma + \epsilon_3$
$y_4 = \phi - \gamma + \epsilon_4$,
where ϵ_i are uncorrelated having mean 0 and variance σ^2.
(a) Show that $\gamma - \phi$ is estimable. What is its BLUE?
(b) Find the residual sum of squares. What is its number of degrees of freedom? [11]

4. Let $\mathbf{X} = (X_1, X_2, X_3, X_4)'$ have mean $\mathbf{0}$ and covariance matrix $\sigma^2\left\{(1 - a^2)I_4 + a^2\mathbf{1}\mathbf{1}'\right\}$, for some $0 < |a| < 1$ and where $\mathbf{1}$ is the vector with all elements equal to 1. Find the partial correlations, $\rho_{12.3}$ and $\rho_{12.34}$. [10]

5. Consider the one-way model:

$$y_{ij} = \mu_i + \epsilon_{ij},\ 1 \le j \le n_i;\ 1 \le i \le k,$$

where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$, $k \ge 4$ and $n_i > 1$ for all i.
(a) Show that $\mu_1 - \mu_2$ is estimable.
(b) What are the methods due to Scheffe and Bonferroni for constructing simultaneous confidence sets for a set of estimable linear functions of μ_i's?
(c) Construct a $100(1 - \alpha)\%$ simultaneous confidence set for $(\mu_1 - \mu_2, \mu_2 - \mu_3, \mu_3 - \mu_4)$. [12]